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Cohen–Macaulay local rings of dimension two and an extended version of a conjecture of J. Sally

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Dedicated to the memory of Professor Hideyuki Matsumura

Abstract

In this paper we prove an extended version of a conjecture of J. Sally. Let $(A,\mathcal{M})$ be a Cohen–Macaulay local ring of dimension d, multiplicity e and embedding codimension h. If the initial degree of A is bigger than or equal to t and $e=\binom{h+t-1}{h}+1$, we prove that the depth of the associated graded ring of A is at least $d-1$ and the h-vector of A has no negative components. The conjecture of Sally was dealing with the case $t=2$ and was proved by these authors in a previous paper. Some new formulas relating certain numerical characters of a two-dimensional Cohen–Macaulay local ring are also given. © 1997 Published by Elsevier Science B.V.

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0. Introduction

Let $(A,\mathcal{M})$ be a local Cohen–Macaulay ring of dimension d, embedding dimension N and multiplicity e. By a classical result of Abhyankar (see [1]), we have $e\geq N-d+1$ and if equality holds, the structure of the associated graded ring $G:=\bigoplus_{n\geq 0}(\mathcal{M}^n/\mathcal{M}^{n+1})$ is well understood and G itself is Cohen–Macaulay (see [10]). In the case $e=N-d+2$, J. Sally proved in [12] that G is not necessarily Cohen–Macaulay, the exceptions being the Cohen–Macaulay local rings of maximal type $e-2$. The main open question there was about the possible depths of G and in fact Sally made the conjecture that $depth(G)\geq d-1$ and gave strong evidence for this to be true.

In [9] we proved this conjecture and also described all the possible Hilbert functions of A.

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But very often we know that a given algebraic variety does not lie on a quadric or more generally does not lie on any hypersurface of degree less than a certain integer t. In this case the bound given by Abhyankar is no more sharp, and in [4] the right bound, involving the initial degree t, is given and the extremal case is studied. It turns out that if we let $h := N - d$, then $e \geq \binom{h+t-1}{h}$ and, if equality holds, A has maximal Hilbert function and G is Cohen–Macaulay. If instead we have $e = \binom{h+t-1}{h} + 1$, then G is not necessarily Cohen–Macaulay (see the examples at the end of the paper), the possible exceptions being again the Cohen–Macaulay local rings of maximal type $\binom{h+t-2}{t-1}$ (see [8, Theorem 3.10]).

In Section 3 we prove that, also in this more general setting, $depth(G) \geq d - 1$ and describe all the possible Hilbert functions of A. This gives a complete solution to the extended version of Sally's conjecture we refer in the title. We discussed this problem in [2] where the guess was formulated.

To us, one of the more interesting aspects of our proof concerns what our methods show about the relationship between certain numerical characters of a Cohen–Macaulay local ring of dimension two, a topic to which Section 2 is devoted.

1. Preliminaries

Let $(A, \mathcal{M})$ be a local ring of dimension d, multiplicity e and residue field $k = A/\mathcal{M}$. The Hilbert function of A is by definition the Hilbert function of the associated graded ring of A which is the homogeneous k-algebra

$$G := gr_{\mathcal{M}}(A) = \bigoplus_{n \geq 0} \mathcal{M}^n / \mathcal{M}^{n+1}.$$

Hence,

$$H_A(n) = H_G(n) = dim_k(\mathcal{M}^n / \mathcal{M}^{n+1}).$$

The generating function of this numerical function is the power series

$$P_A(z) = \sum_{n \in \mathbb{N}} H_A(n) z^n$$

which is called the Hilbert Series of A. This series is rational and there exists a polynomial $h(z) \in \mathbf{Z}[z]$ such that

$$P_A(z) = \frac{h(z)}{(1-z)^d},$$

where $h(1) = e \geq 1$.

The polynomial $h(z) = h_0 + h_1 z + \cdots + h_s z^s$ is called the h-polynomial of A and the vector $(h_0, h_1, \ldots, h_s)$ the h-vector of A.

For every $i \geq 0$, we let

$$e_i := \frac{h^{(i)}(1)}{i!}$$

and

$$\binom{X+i}{i} := \frac{(X+i)\cdots(X+1)}{i!}.$$

Then

$$e_0 = e$$

and the polynomial

$$p_A(X) := \sum_{i=0}^{d-1}(-1)^i e_i \binom{X+d-i-1}{d-i-1}$$

has rational coefficients and degree $d-1$; further for every $n \gg 0$

$$p_A(n) = H_A(n).$$

The polynomial $p_A(X)$ is called the Hilbert polynomial of A.

The embedding codimension of A is the integer

$$h := embcod(A) := H_A(1) - d.$$

It is clear that $h = h_1$, the coefficient of z in the h-polynomial of A. Further $embcod(A) = 0$ if and only if A is a regular local ring.

We denote by $indeg(A)$ the initial degree of A which is the integer defined as

$$indeg(A) = \min\left\{ j : H_A(j) < \binom{H_A(1)+j-1}{j} \right\}.$$

If a is an element in $\mathcal{M}$, $a \notin \mathcal{M}^2$, then $H_A(1) = H_{A/aA}(1) + 1$ and we may write $\mathcal{M} = (a_1, \ldots, a_n)$ where $a = a_1$ and $n = H_A(1)$. Then the associated graded ring of A can be presented as

$$G = k[X_1, \ldots, X_n]/I,$$

where I is the homogeneous ideal generated by the forms F of degree i such that

$$F(a_1, \ldots, a_n) \in \mathcal{M}^{i+1}.$$

If $indeg(A) \geq t$, then $I_j = 0$ for every $j \leq t-1$ and we claim that

$$\mathcal{M}^{i+1} : a = \mathcal{M}^i \;\forall i = 0 \ldots, t-1. \tag{1}$$

Since $\mathcal{M}^2 : a = \mathcal{M}$, this is clear if $t \leq 2$; if $t \geq 3$, we may assume, by contradiction, that $\mathcal{M}^i : a = \mathcal{M}^{i-1}$ and $\mathcal{M}^{i+1} : a \neq \mathcal{M}^i$ for some i, $2 \leq i \leq t-1$. Let $b \notin \mathcal{M}^i$ be an element such that $ab \in \mathcal{M}^{i+1}$. Then $b \in \mathcal{M}^{i-1}$ so that $b = F(a_1, \ldots, a_n)$ where $F \in k[X_1, \ldots, X_n]$ is a form of degree $i-1$. It follows that

$$aF(a_1, \ldots, a_n) = ab \in \mathcal{M}^{i+1},$$

hence $0 \neq X_1 F \in I_i$, which is a contradiction. This proves the claim.

Now we recall a classical result of Singh [13] which asserts that for every $a \in \mathcal{M}$ and for every $i \geq 0$

$$H_A(i) = \sum_{j=0}^{i} H_{A/aA}(j) - \lambda(\mathcal{M}^{i+1} : a/\mathcal{M}^i),$$

where $\lambda(M)$ denotes the length of an A-module M.

Using this equality and (1), we easily get that for every $a \in \mathcal{M}$, $a \notin \mathcal{M}^2$

$$indeg(A/aA) \geq indeg(A). \tag{2}$$

We recall that if A has positive dimension, an element x in $\mathcal{M}$ is called superficial for A if there exists an integer $c > 0$ such that

$$(\mathcal{M}^n : x) \cap \mathcal{M}^c = \mathcal{M}^{n-1}$$

for every $n > c$.

It is easy to see that a superficial element x is not in $\mathcal{M}^2$ and that x is superficial for A if and only if $x^* := \overline{x} \in \mathcal{M}/\mathcal{M}^2$ does not belong to the relevant associated primes of G. Hence, if the residue field is infinite, superficial elements always exist.

Further if A has positive depth, every superficial element is also a regular element in A.

A sequence $x_1, \ldots, x_r$ in the local ring $(A, \mathcal{M})$ is called a superficial sequence for A, if x_1 is superficial for A and $\overline{x_i}$ is superficial for $A/(x_1, \ldots, x_{i-1})$ for $2 \leq i \leq r$.

By passing, if needed, to the local ring $A[X]_{(\mathcal{M},X)}$ we may assume that the residue field is infinite. Hence if $depth(A) \geq r$, every superficial sequence $x_1, \ldots, x_r$ is also a regular sequence in A. Such a sequence has the right properties for a good behaviour of the numerical invariants under reduction modulo the ideal it generates.

In particular if $J = (x_1, \ldots, x_r)$, and $(B, \mathcal{N}) = (A/J, \mathcal{M}/J)$, then B is a local ring with

- $dim(B) = d - r$,
- If $depth(A) \geq r$, then $depth(B) = depth(A) - r$,
- $embcod(A) = embcod(B)$,
- $indeg(B) \geq indeg(A)$,
- $e_i(A) = e_i(B)$ for $i = 0, \ldots, d - r$.

The following relevant properties of superficial sequences will also be needed.

- $depth(gr_{\mathcal{M}}(A)) \geq r \Leftrightarrow x_1^*, \ldots, x_r^*$ is a regular sequence in $gr_{\mathcal{M}}(A) \Leftrightarrow P_A(z) = P_B(z)/(1-z)^r \Leftrightarrow \mathcal{M}^j \cap J = J\mathcal{M}^{j-1}$ for every $j \geq 1$.
- $depth(gr_{\mathcal{M}}(A)) \geq r + 1 \Leftrightarrow depth(gr_{\mathcal{N}}(B)) \geq 1$.

This last property is the so-called *Sally's machine* which is a very important trick to reduce dimension in questions relating to depth properties of $gr_{\mathcal{M}}(A)$. Sally proved this result in the case $r = d - 1$ in [11]; a complete and nice proof of the general case can be found in Lemma 2.2 of [5].

2. Two dimensional Cohen–Macaulay local rings

In this section we collect some results which relate certain numerical invariants of a Cohen–Macaulay local ring of dimension two.

We first recall that in the case A is a one-dimensional Cohen–Macaulay local ring and x a superficial element, from the diagram

$$\begin{array}{ccccc} A & \supset & \mathcal{M}^j & \supset & \mathcal{M}^{j+1} \\ \cup & & \cup & & \cup \\ xA & \supset & x\mathcal{M}^j & = & x\mathcal{M}^j \end{array}$$

we get

$$H_A(j) = \lambda(\mathcal{M}^j/x\mathcal{M}^j) - \lambda(\mathcal{M}^{j+1}/x\mathcal{M}^j).$$

But $\lambda(A/xA) = e$ and $A/\mathcal{M}^j \simeq xA/x\mathcal{M}^j$ so that for every $j \geq 0$

$$H_A(j) = e - \lambda(\mathcal{M}^{j+1}/x\mathcal{M}^j). \tag{3}$$

If we let $\sigma_j := \lambda(\mathcal{M}^{j+1}/x\mathcal{M}^j)$, we have $\sigma_0 = e - H_A(0) = e - 1$, and if s is the degree of the h-polynomial of A, then $\sigma_j = 0$ for every $j \geq s$ and

$$P_A(z) = \frac{1 + (\sigma_0 - \sigma_1)z + (\sigma_1 - \sigma_2)z^2 + \cdots + (\sigma_{s-2} - \sigma_{s-1})z^{s-1} + \sigma_{s-1}z^s}{1 - z}.$$

This clearly implies

$$e_1 = \sum_{j=0}^{s-1} \sigma_j$$

and

$$e_2 = \sum_{j=1}^{s-1} j\sigma_j.$$

Similar formulas can be found in the two-dimensional case. We need new numerical invariants which have been introduced by Huneke in [6]. Since we are assuming A to be Cohen–Macaulay, we can find in $\mathcal{M}$ a superficial sequence x, y and we let $J = (x, y)$ be the ideal they generate. The main point in Huneke's result is the fact that if $\mathcal{I} \supseteq \mathcal{P}$ are ideals in A then we have a short exact sequence:

$$0 \to \mathcal{I} : J/\mathcal{P} : J \to (\mathcal{I}/\mathcal{P})^2 \to J\mathcal{I}/J\mathcal{P} \to 0. \tag{4}$$

For every integer $n \geq 1$ we let

$$v_n := \lambda(\mathcal{M}^{n+1}/J\mathcal{M}^n) - \lambda(\mathcal{M}^n : J/\mathcal{M}^{n-1})$$

and for $n = 0$ we let

$$v_0 := e - 1.$$

By using (4), Huneke proved that for every $n \geq 1$

$$H_A(n) - H_A(n-1) = e - v_n. \tag{5}$$

Let s be the degree of the h-polynomial of A; since the h-polynomial of a two-dimensional local ring is the second difference of its Hilbert function, we must have $v_j = 0$ for every $j \geq s$ and

$$P_A(z) = \frac{1 + (v_0 - v_1)z + (v_1 - v_2)z^2 + \cdots + (v_{s-2} - v_{s-1})z^{s-1} + v_{s-1}z^s}{(1-z)^2}.$$

This gives

$$e_1 = \sum_{j=0}^{s-1} v_j \tag{6}$$

and

$$e_2 = \sum_{j=1}^{s-1} j v_j. \tag{7}$$

Unfortunately the integers v_i can be negative; however, the following construction due to Ratliff and Rush (see [7]), gives a way to overcome the problem.

Let $(A, \mathcal{M})$ be a Cohen–Macaulay local ring. For every n we consider the chain of ideals

$$\mathcal{M}^n \subseteq \mathcal{M}^{n+1} : \mathcal{M} \subseteq \mathcal{M}^{n+2} : \mathcal{M}^2 \subseteq \cdots \subseteq \mathcal{M}^{n+k} : \mathcal{M}^k \subseteq \cdots$$

This chain stabilizes at an ideal which was denoted by Ratliff and Rush as

$$\widetilde{\mathcal{M}^n} := \bigcup_{k \geq 1} (\mathcal{M}^{n+k} : \mathcal{M}^k).$$

We have

$$\widetilde{\mathcal{M}} = \mathcal{M},$$

and for every i, j

$$\mathcal{M}^i \subseteq \widetilde{\mathcal{M}^i}, \qquad \widetilde{\mathcal{M}^i}\widetilde{\mathcal{M}^j} \subseteq \widetilde{\mathcal{M}^{i+j}}.$$

Further, if x is superficial for A,

$$\widetilde{\mathcal{M}^{n+1}} : x = \widetilde{\mathcal{M}^n}$$

for every $n \geq 0$.

We define for every $n \geq 0$

$$\rho_n := \lambda(\widetilde{\mathcal{M}^{n+1}} / J\widetilde{\mathcal{M}^n}).$$

For example, we have

$$\rho_0 = e - 1.$$

We will make use of the fact that the usual Rees algebra $\mathscr{R}(\mathscr{M})$ is a subalgebra of the Rees algebra associated to the Ratliff–Rush filtration, namely

$$\bigoplus_{n\geq 0} \mathscr{M}^n T^n \subseteq \bigoplus_{n\geq 0} \widetilde{\mathscr{M}^n} T^n.$$

This implies that $\bigoplus_{n\geq 0} \widetilde{\mathscr{M}^n}/\mathscr{M}^n$ has a canonical structure as a graded module over $\mathscr{R}(\mathscr{M})$.

It is thus natural to introduce a new set of numerical invariants, namely to let for every $n \geq 0$

$$a_n := \lambda(\widetilde{\mathscr{M}^n}/\mathscr{M}^n).$$

In particular, we have

$$a_0 = a_1 = 0.$$

The following proposition clarifies the relationship between all these integers.

Proposition 2.1. *Let $(A, \mathscr{M})$ be a two-dimensional Cohen–Macaulay local ring. For every $n \geq 1$, we have*

$$\rho_n + 2a_n = a_{n-1} + a_{n+1} + v_n.$$

Proof. By letting $\mathscr{I} = \widetilde{\mathscr{M}^n}$ and $\mathscr{P} = \mathscr{M}^n$ in (4), we get for every $n \geq 1$,

$$2a_n = \lambda(\widetilde{\mathscr{M}^n} : J/\mathscr{M}^n : J) + \lambda(J\widetilde{\mathscr{M}^n}/J\mathscr{M}^n).$$

Since

$$\widetilde{\mathscr{M}^{n-1}} \subseteq \widetilde{\mathscr{M}^n} : J \subseteq \widetilde{\mathscr{M}^n} : x = \widetilde{\mathscr{M}^{n-1}} \supseteq \mathscr{M}^n : J \supseteq \mathscr{M}^{n-1},$$

we get

$$\lambda(J\widetilde{\mathscr{M}^n}/J\mathscr{M}^n) = 2a_n - a_{n-1} + \lambda(\mathscr{M}^n : J/\mathscr{M}^{n-1}).$$

On the other hand, by the diagram

$$\begin{array}{ccc} \widetilde{\mathscr{M}^{n+1}} & \supset & J\widetilde{\mathscr{M}^n} \\ \cup & & \cup \\ \mathscr{M}^{n+1} & \supset & J\mathscr{M}^n, \end{array}$$

we get

$$\rho_n + \lambda(J\widetilde{\mathscr{M}^n}/J\mathscr{M}^n) = a_{n+1} + \lambda(\mathscr{M}^{n+1}/J\mathscr{M}^n).$$

It follows that

$$\rho_n + 2a_n - a_{n-1} + \lambda(\mathcal{M}^n : J/\mathcal{M}^{n-1}) = a_{n+1} + \lambda(\mathcal{M}^{n+1}/J\mathcal{M}^n),$$

hence

$$\rho_n + 2a_n = a_{n-1} + a_{n+1} + v_n. \qquad \square$$

As a trivial application of this formula we get a way to write e_1 and e_2 as sums of non-negative integers.

It is clear that there exists an integer r such that $a_n = 0$ for $n \geq r$. By the proposition this implies $\rho_n = v_n$ for every $n \geq r+1$ and further, by some easy computation, $\sum_{j=0}^{r} v_j = \sum_{j=0}^{r} \rho_j$ and $\sum_{j=1}^{r} jv_j = \sum_{j=1}^{r} j\rho_j$. By using (6) and (7) we get

$$e_1 = \sum_{j \geq 0} \rho_j \tag{8}$$

and

$$e_2 = \sum_{j \geq 1} j\rho_j. \tag{9}$$

These formulas have been proved in [5, Corollary 4.13] by using a deeper homological approach.

From the above proposition, since $a_0 = a_1 = 0$, we get $\rho_1 = a_2 + v_1$ and by induction

$$a_n = \sum_{j=1}^{n-1} (n-j)(\rho_j - v_j), \quad \forall n \geq 2. \tag{10}$$

In the following we need to control the behaviour of the integers ρ_i and v_i for $i = 0, \ldots, indeg(A) - 1$.

Proposition 2.2. *Let $(A, \mathcal{M})$ be a two-dimensional Cohen–Macaulay local ring with $indeg(A) \geq t$. For every $n = 0, \ldots, t-1$ we have*
- $\mathcal{M}^{n+1} \cap J = J\mathcal{M}^n$.
- $v_n = \lambda(\mathcal{M}^{n+1}/J\mathcal{M}^n)$.
- $\rho_n - v_n = \lambda(\widetilde{\mathcal{M}^{n+1}}/J\widetilde{\mathcal{M}^n} + \mathcal{M}^{n+1})$.

Proof. Let $ax + by \in \mathcal{M}^{n+1}$ with $a, b \in A$. Then $by \in \mathcal{M}^{n+1} + (x)$ hence by (1) $b \in \mathcal{M}^n + (x)$. Thus, we can write $b = cx + d$ with $d \in \mathcal{M}^n$. Hence, we get $cy + a \in \mathcal{M}^{n+1} : x = \mathcal{M}^n$ and $a = -cy + e$ with $e \in \mathcal{M}^n$. It follows

$$ax + by = ex + dy \in J\mathcal{M}^n.$$

This proves the first assertion.

As for the second one, we remark that for every $j = 0, \ldots, t-1$, we have by (1) $\mathcal{M}^{j+1} : x = \mathcal{M}^j$, so that for every $n = 1, \ldots, t$

$$\mathcal{M}^{n-1} \subseteq \mathcal{M}^n : J \subseteq \mathcal{M}^n : x = \mathcal{M}^{n-1}.$$

This implies

$$\mathcal{M}^n : J = \mathcal{M}^{n-1}$$

and $v_n = \lambda(\mathcal{M}^{n+1}/J\mathcal{M}^n)$.

Finally, we have

$$J\widetilde{\mathcal{M}^n} \subseteq J\widetilde{\mathcal{M}^n} + \mathcal{M}^{n+1} \subseteq \widetilde{\mathcal{M}^{n-1}};$$

hence

$$\begin{aligned}\lambda(\widetilde{\mathcal{M}^{n+1}}/J\widetilde{\mathcal{M}^n} + \mathcal{M}^{n+1}) &= \rho_n - \lambda(J\widetilde{\mathcal{M}^n} + \mathcal{M}^{n+1}/J\widetilde{\mathcal{M}^n}) \\ &= \rho_n - \lambda(\mathcal{M}^{n+1}/J\widetilde{\mathcal{M}^n} \cap \mathcal{M}^{n+1}).\end{aligned}$$

Since

$$J\mathcal{M}^n \subseteq J\widetilde{\mathcal{M}^n} \cap \mathcal{M}^{n+1} \subseteq \mathcal{M}^{n+1} \cap J = J\mathcal{M}^n,$$

we get $J\mathcal{M}^n = J\widetilde{\mathcal{M}^n} \cap \mathcal{M}^{n+1}$ and the conclusion follows. □

3. The main theorem

In this section we give a proof of an extended version of a conjecture of J. Sally.

Theorem 3.1. *Let $(A, \mathcal{M})$ be a d-dimensional Cohen–Macaulay local ring and t an integer, $t \geq 2$. The following conditions are equivalent:*

- *$indeg(A) \geq t$ and $e = \binom{h+t-1}{h} + 1$.*
- *There exists an integer s, $t \leq s \leq \binom{h+t-1}{h}$, such that*

$$P_A(z) = \frac{\sum_{i=0}^{t-1} \binom{h+i-1}{i} z^i + z^s}{(1-z)^d}.$$

If either of the above conditions holds, then $depth(G) \geq d-1$ and G is Cohen–Macaulay if and only if $s = t$.

We start by proving this in the one-dimensional case. First we need a couple of easy results which will be used also later.

Lemma 3.2. *Let A be a ring, I and K ideals in A. If $t \geq 2$ is an integer such that $K \subseteq I^t$ and $\lambda(I^t/K) = 1$, then either $I^{t+1} = IK$ or $I^t = K + (a^t)$ for some $a \in I$.*

Proof. Let $I = (a_1, \ldots, a_r)$; if $a_i I^{t-1} \subseteq K$ for every $i = 1, \ldots, r$, then $I^t \subseteq K$, a contradiction. Hence let $a := a_1$ and $aI^{t-1} \not\subseteq K$. If $I^{t+1} \neq IK$, we claim that $a^t \notin K$, which gives the conclusion. To prove the claim we show that if with $2 \leq j \leq t$ we have

$a^jI^{t-j} \subseteq K$ then $a^{j-1}I^{t-j+1} \subseteq K$. Let us assume by contradiction that $a^{j-1}b \notin K$ for some $b \in I^{t-j+1}$; then $I^t = K + (a^{j-1}b)$ so that

$$I^{t+1} = IK + a^{j-1}bI \subseteq IK + aI^t = IK + aK + (a^jb) \subseteq IK + a^jI^{t-j+1} \subseteq IK.$$

The conclusion follows. □

Since in the following A is Cohen–Macaulay, we can find a maximal superficial sequence in A and we denote as usual by J the ideal it generates.

Proposition 3.3. *Let $(A, \mathcal{M})$ be a local Cohen–Macaulay ring of dimension one or two such that $indeg(A) \geq t \geq 2$ and $e = \binom{h+t-1}{h} + 1$. The following conditions hold:*

1. $v_{t-1} = \lambda(\mathcal{M}^t/J\mathcal{M}^{t-1}) = 1$.
2. $\mathcal{M}^{j+2} \subseteq JM^j$ *for every* $j \geq t-1$.
3. *Either* $\mathcal{M}^{t+1} = J\mathcal{M}^t$ *or there exists* $w \in \mathcal{M}$ *such that* $\mathcal{M}^{j+1} = J\mathcal{M}^j + (w^{j+1})$ *for every* $j \geq t-1$.
4. $\lambda(\mathcal{M}^{j+1}/J\mathcal{M}^j) \leq 1$ *for every* $j \geq t-1$.
5. $v_j = \lambda(\mathcal{M}^{j+1}/J\mathcal{M}^j) = e - \binom{h+j}{j}$ *for every* $j = 0, \ldots, t-2$.

Proof. If $d = 1$, since $indeg(A) \geq t$ we have

$$e - 1 = \binom{h+t-1}{h} = H_A(t-1).$$

Hence, by (3), we get

$$e - 1 = H_A(t-1) = e - \lambda(\mathcal{M}^t/x\mathcal{M}^{t-1}),$$

which implies

$$\lambda(\mathcal{M}^t/x\mathcal{M}^{t-1}) = 1$$

as required.

If $d = 2$, by (5) we have

$$1 = e - H_A(t-1) + H_A(t-2) = v_{t-1} = \lambda(\mathcal{M}^t/J\mathcal{M}^{t-1}) - \lambda(\mathcal{M}^{t-1} : J/\mathcal{M}^{t-2}).$$

Since by (1) $\mathcal{M}^{t-1} : J \subseteq \mathcal{M}^{t-1} : x = \mathcal{M}^{t-2}$, we have

$$\lambda(\mathcal{M}^t/J\mathcal{M}^{t-1}) = 1.$$

This proves 1.

We prove now 2. One has

$$\mathcal{M}^t \supseteq \mathcal{M}^{t+1} + J\mathcal{M}^{t-1} \supseteq J\mathcal{M}^{t-1}$$

and also, if $\mathcal{M}^t = \mathcal{M}^{t+1} + J\mathcal{M}^{t-1}$, by Nakayama $\mathcal{M}^t = J\mathcal{M}^{t-1}$ against 1. Hence,

$$\mathcal{M}^{t+1} \subseteq J\mathcal{M}^{t-1}$$

and the second assertion follows by multiplication by $\mathcal{M}$.

We pass to the third assertion. Since $\lambda(\mathcal{M}^t/J\mathcal{M}^{t-1})=1$, if $\mathcal{M}^{t+1} \neq J\mathcal{M}^t$, by the above lemma there exists an element $w \in \mathcal{M}$ such that

$$\mathcal{M}^t = J\mathcal{M}^{t-1} + (w^t).$$

But if $j \geq t$ and $\mathcal{M}^j = J\mathcal{M}^{j-1} + (w^j)$, we get

$$\begin{aligned}\mathcal{M}^{j+1} &= J\mathcal{M}^j + w^j\mathcal{M} \subseteq J\mathcal{M}^j + w\mathcal{M}^j \\ &= J\mathcal{M}^j + wJ\mathcal{M}^{j-1} + (w^{j+1}) = J\mathcal{M}^j + (w^{j+1}) \subseteq \mathcal{M}^{j+1}.\end{aligned}$$

This proves 3. Since $\mathcal{M}^{j+1}/J\mathcal{M}^j$ are k-vector spaces, the fourth assertion also follows.

We prove now the last assertion. If $d=1$, since $indeg(A) \geq t$, the formula is a consequence of (3). If $d=2$, we need only to apply (5). $\square$

Proposition 3.4. *Let $(A, \mathcal{M})$ be a Cohen–Macaulay local ring of dimension one and t an integer, $t \geq 2$. The following conditions are equivalent:*
- *indeg $(A) \geq t$ and $e = \binom{h+t-1}{h} + 1$.*
- *There exists an integer s such that $t \leq s \leq \binom{h+t-1}{h}$ and*

$$P_A(z) = \frac{\sum_{i=0}^{t-1} \binom{h+i-1}{i} z^i + z^s}{(1-z)}.$$

Proof. We need only to prove that the first condition implies the second. Since $indeg(A) \geq t$, we have

$$H_A(j) = \binom{h+j}{j}$$

for every $j \leq t-1$. By (3) and 4 in the above proposition, we have for every $j \geq t$

$$H_A(j) = e - \lambda(\mathcal{M}^{j+1}/x\mathcal{M}^j) \geq \binom{h+t-1}{h}.$$

If we let s be the least integer such that $H_A(s) = e = \binom{h+t-1}{h} + 1$, then $\mathcal{M}^{s+1} = x\mathcal{M}^s$, so that $\mathcal{M}^{r+1} = x\mathcal{M}^r$ and $H_A(r) = e$ for every $r \geq s$. This proves that

$$P_A(z) = \frac{\sum_{i=0}^{t-1} \binom{h+i-1}{i} z^i + z^s}{(1-z)}.$$

By the well-known theorem of Macaulay which characterizes the Hilbert functions of standard graded algebras, $H_A(e-1) = e$ so that $s \leq e-1 = \binom{h+t-1}{h}$.

This gives the conclusion. $\square$

We come back now to the general case of the theorem. First of all we have $s=t$ if and only if the h-vector of A coincides with that of its artinian reduction. Hence, the last assertion on the Cohen–Macaulayness of G is clear.

Further if

$$P_A(z)=\frac{\sum_{i=0}^{t-1}\binom{h+i-1}{i}z^i+z^s}{(1-z)^d},$$

then $e=\binom{h+t-1}{h}+1$ and $indeg(A)\geq t$. The converse holds easily if $d=0$, while if $d=1$, it follows by the above proposition.

If $d\geq 2$, we let

$$B:=A/(x_1,\ldots,x_{d-2})$$

and

$$C:=A/(x_1,\ldots,x_{d-1}),$$

where $x_1,\ldots,x_{d-1}$ is a superficial sequence in A. We have $dim(C)=1$, $dim(B)=2$,

$$\binom{h+t-1}{h}+1=e(A)=e(B)=e(C)$$

and by (2)

$$indeg(C)\geq indeg(B)\geq indeg(A)\geq t.$$

Hence, by Proposition 3.4,

$$P_C(z)=\frac{\sum_{i=0}^{t-1}\binom{h+i-1}{i}z^i+z^s}{(1-z)}.$$

If we can prove that the depth of the associated graded ring of B is positive, then, by using Sally's machine, we get $depth(G)\geq 1+d-2=d-1$. This implies

$$P_A(z)=\frac{P_C(z)}{(1-z)^{d-1}}=\frac{\sum_{i=0}^{t-1}\binom{h+i-1}{i}z^i+z^s}{(1-z)^d}$$

and the conclusion of the theorem follows.

Henceforth, we may assume $dim(A)=2$ and, by the above remark, we need to prove that $depth(G)\geq 1$. As before, we let $J=(x,y)$ be the ideal generated by a superficial sequence and $R:=A/xA$.

Since R is now a one-dimensional Cohen–Macaulay local ring with $e=\binom{h+t-1}{h}+1$ and $indeg(R)\geq t$, by Proposition 3.4 and (3), we have

$$\begin{aligned}e_1(R)&=\sum_{j=0}^{s-1}\sigma_j=\sum_{j=0}^{t-1}(e-H_A(j))+\sum_{j=t}^{s-1}(e-H_A(j))\\&=te-\sum_{j=0}^{t-1}\binom{h+j}{j}+s-t=te+s-t-\binom{h+t}{h+1}.\end{aligned}$$

Further, since $d=2$,

$$e_1(A)=e_1(R)=te+s-t-\binom{h+t}{h+1}.$$

Proposition 3.5. *With the above notation the following conditions hold*:
- $\lambda(\mathcal{M}^{j+1}/J\mathcal{M}^j)=1$ *for every* $j=t-1,\dots,s-1$.
- $\mathcal{M}^{j+1}:x=\mathcal{M}^j$ *for every* $j=0,\dots,s-1$.
- $v_j=1$ *for every* $j=t-1,\dots,s-1$.
- $depth(G)>0 \Leftrightarrow \mathcal{M}^{s+1}=J\mathcal{M}^s$.

Proof. From the proof of Proposition 3.4, we have

$$\lambda((\mathcal{M}/x)^{j+1}/y(\mathcal{M}/x)^j)=\begin{cases}1 & \text{if } t-1\le j\le s-1,\\ 0 & \text{if } j\ge s.\end{cases}$$

It is easy to see and proved in [9] that for every $j\ge 0$ there is an exact sequence

$$0\to \mathcal{M}^j:x/\mathcal{M}^j:J \xrightarrow{y} \mathcal{M}^{j+1}:x/\mathcal{M}^j \xrightarrow{x} \mathcal{M}^{j+1}/J\mathcal{M}^j \to (\mathcal{M}/x)^{j+1}/y(\mathcal{M}/x)^j \to 0.$$

Thus, if $t-1\le j\le s-1$, since $\lambda(\mathcal{M}^{j+1}/J\mathcal{M}^j)\le 1$, the first assertion follows from the above exact sequence.

The second property follows by (1) if $j\le t-1$. Let $j\ge t$; since

$$\mathcal{M}^j:x\supseteq \mathcal{M}^j:J\supseteq \mathcal{M}^{j-1},$$

by induction on j the first module on the left in the above exact sequence is 0, hence the second one is zero too since the last two modules share the same length.

The third assertion is now a trivial consequence of the first two.

Finally if $depth(G)>0$, then x^* is a regular element in G, hence $\mathcal{M}^{s+1}:x=\mathcal{M}^s$. Since if $j=s$ the last module in the above exact sequence is zero, we get $\mathcal{M}^{s+1}=J\mathcal{M}^s$.

Conversely, let $\mathcal{M}^{s+1}=J\mathcal{M}^s$. Since

$$\mathcal{M}^{s-1}=\mathcal{M}^s:x\supseteq \mathcal{M}^s:J\supseteq \mathcal{M}^{s-1},$$

we get by the above exact sequence $\mathcal{M}^{s+1}:x=\mathcal{M}^s$. Since $\mathcal{M}^{j+1}=J\mathcal{M}^j$ for every $j\ge s$, we can go on and finally prove that

$$\mathcal{M}^{j+1}:x=\mathcal{M}^j$$

for every $j\ge s$. Hence, by the second assertion, $\mathcal{M}^{j+1}:x=\mathcal{M}^j$ for every $j\ge 0$ and this implies $depth(G)>0$, as desired. □

We will need the following result which has been proved in [9] and which is the crucial point in the proof of the theorem. We do not insert here a proof; we only remark that, as we said before, $\bigoplus(\widetilde{\mathcal{M}^n}/\mathcal{M}^n)$ is a graded module over the Rees algebra $\mathcal{R}(\mathcal{M})$. By using this and the standard trick as in the classical Cayley–Hamilton theorem, one gets the conclusion.

Proposition 3.6. *Let* $(A,\mathcal{M})$ *be a local ring,* $p\ge 2$ *an integer and* $J\subseteq \mathcal{M}$ *an ideal of* A. *For every integer* $n=2,\dots,p$ *suppose we are given ideals*

$$I_n=(a_{1n},\dots,a_{v_n n})\subseteq \widetilde{\mathcal{M}^n}.$$

Let w be an element in $\mathcal{M}$ such that

$$\begin{aligned} wI_2 &\subseteq JI_2 + I_3 + \mathcal{M}^3, \\ wI_3 &\subseteq J^2I_2 + JI_3 + I_4 + \mathcal{M}^4, \\ &\vdots \\ wI_{p-1} &\subseteq J^{p-2}I_2 + J^{p-3}I_3 + \cdots + JI_{p-1} + I_p + \mathcal{M}^p, \\ wI_p &\subseteq J^{p-1}I_2 + J^{p-2}I_3 + \cdots + JI_p + \mathcal{M}^{p+1}. \end{aligned}$$

If we let $v := \sum_{i=2}^{p} v_i$, then there exists an element $\sigma \in J\mathcal{M}^{v-1}$ such that for every $n = 2, \ldots, p$ and $i = 1, \ldots, v_n$

$$w^v a_{in} \equiv \sigma a_{in} \mod \mathcal{M}^{v+n}.$$

We can finish now the proof of the theorem.

Theorem 3.7. *Let $(A, \mathcal{M})$ be a Cohen–Macaulay local ring of dimension two such that $indeg(A) \geq t$ and $e = \binom{h+t-1}{h} + 1$. Then*

$$depth(G) \geq 1.$$

Proof. For every $n \geq 2$ we have

$$\widetilde{\mathcal{M}^n} \supseteq J\widetilde{\mathcal{M}^{n-1}} + \mathcal{M}^n \supseteq J\widetilde{\mathcal{M}^{n-1}}$$

and

$$\lambda(\widetilde{\mathcal{M}^n}/J\widetilde{\mathcal{M}^{n-1}}) = \rho_{n-1}.$$

Hence,

$$\lambda(\widetilde{\mathcal{M}^n}/J\widetilde{\mathcal{M}^{n-1}} + \mathcal{M}^n) \leq \rho_{n-1}$$

and equality holds if and only if

$$\mathcal{M}^n \subseteq J\widetilde{\mathcal{M}^{n-1}}.$$

Further we can find elements $a_{1n}, \ldots, a_{v_n n} \in \widetilde{\mathcal{M}^n}$ such that their residue classes modulo $J\widetilde{\mathcal{M}^{n-1}} + \mathcal{M}^n$ form a minimal system of generators of the module $\widetilde{\mathcal{M}^n}/J\widetilde{\mathcal{M}^{n-1}} + \mathcal{M}^n$. It is clear that we have

$$v_n \leq \lambda(\widetilde{\mathcal{M}^n}/J\widetilde{\mathcal{M}^{n-1}} + \mathcal{M}^n) \leq \rho_{n-1}$$

and if $\mathcal{M}^n \not\subseteq J\widetilde{\mathcal{M}^{n-1}}$, then

$$v_n < \rho_{n-1}. \tag{11}$$

If $n \leq t$, by Proposition 2.2, we can be more precise, namely

$$v_n \leq \rho_{n-1} - v_{n-1}.$$

If we let

$$I_n := (a_{1n}, \ldots, a_{v_n n}),$$

then $I_n \subseteq \widetilde{\mathcal{M}^n}$ and

$$\widetilde{\mathcal{M}^n} = J\widetilde{\mathcal{M}^{n-1}} + \mathcal{M}^n + I_n.$$

Since

$$\widetilde{\mathcal{M}^2} = J\widetilde{\mathcal{M}} + \mathcal{M}^2 + I_2 = \mathcal{M}^2 + I_2,$$

we get

$$\widetilde{\mathcal{M}^3} = J\widetilde{\mathcal{M}^2} + \mathcal{M}^3 + I_3 = J\mathcal{M}^2 + JI_2 + \mathcal{M}^3 + I_3 = JI_2 + I_3 + \mathcal{M}^3.$$

Going on in this way, we obtain for every $r \geq 2$

$$\widetilde{\mathcal{M}^r} = \sum_{j=2}^{r} J^{r-j} I_j + \mathcal{M}^r. \tag{12}$$

Now we recall that by Proposition 2.2 and the last assertion in Proposition 3.3, for every $j \leq t-2$, we have

$$\rho_j \geq v_j = e - \binom{h+j}{j},$$

hence

$$\begin{aligned}
et - \binom{h+t}{h+1} + s - t = e_1 &= \sum_{j\geq 0} \rho_j = \sum_{j=0}^{t-2} \rho_j + \sum_{j\geq t-1} \rho_j \\
&\geq \sum_{j=0}^{t-2} \left(e - \binom{h+j}{j} \right) + \sum_{j\geq t-1} \rho_j = (t-1)e - \binom{h+t-1}{t-2} + \sum_{j\geq t-1} \rho_j \\
&= (t-1)e - \binom{h+t}{h+1} + \binom{h+t-1}{h} + \sum_{j\geq t-1} \rho_j \\
&= et - \binom{h+t}{h+1} - 1 + \sum_{j\geq t-1} \rho_j.
\end{aligned}$$

It follows that

$$\sum_{j\geq t-1} \rho_j \leq s - t + 1.$$

We distinguish two cases:

(i) $\rho_{t-1} = \cdots = \rho_{s-1} = 1$. With this assumption the above inequality turns out to be an equality and this implies

$$\rho_j = v_j$$

for every $j \le t-2$ and

$$\rho_s = 0.$$

Further by Proposition 3.5, we have $1 = v_j = \rho_j$ for $t-1 \le j \le s-1$. By (10), this gives $a_n = 0$ for every $n \le s$, hence

$$\mathcal{M}^{s+1} \subseteq \widetilde{\mathcal{M}^{s+1}} = J\widetilde{\mathcal{M}^s} = J\mathcal{M}^s \subseteq \mathcal{M}^{s+1}$$

which implies

$$\mathcal{M}^{s+1} = J\mathcal{M}^s$$

so that, by Proposition 3.5, depth$(G) \ge 1$.

(ii) There exists an integer j such that $t-1 \le j \le s-1$ and $\rho_j \ne 1$. In this case, since by Proposition 2.2 and 3.3 $\rho_{t-1} \ge v_{t-1} = 1$, the condition $\sum_{j \ge t-1} \rho_j \le s-t+1$ implies $s \ge t+1$ and $\rho_j = 0$ for some $t \le j \le s-1$. Thus, we may assume

$$\rho_{t-1}, \dots, \rho_{p-1} \ne 0, \quad \rho_p = 0$$

with $t \le p \le s-1$.

We also let k be the least integer n such that $\mathcal{M}^{n+1} \subseteq J\widetilde{\mathcal{M}^n}$.

We remark that

$$\mathcal{M}^t \not\subseteq J\widetilde{\mathcal{M}^{t-1}}$$

otherwise by Proposition 2.2, $\mathcal{M}^t \subseteq J \cap \mathcal{M}^t = J\mathcal{M}^{t-1}$, which contradicts the equality $\lambda(\mathcal{M}^t/J\mathcal{M}^{t-1}) = 1$.

Further, since $\rho_p = 0$, $\mathcal{M}^{p+1} \subseteq \widetilde{\mathcal{M}^{p+1}} = J\widetilde{\mathcal{M}^p}$, hence we have

$$t \le k \le p \le s-1.$$

By the true definition of k and (11), we have for every $j = t, \dots, k$

$$v_j < \rho_{j-1} \tag{13}$$

and for every $j \le t-1$

$$v_j \le \rho_{j-1} - v_{j-1}.$$

We refer now to Proposition 3.3. If $\mathcal{M}^{t+1} = J\mathcal{M}^t$, then $\mathcal{M}^{s+1} = J\mathcal{M}^s$, and *depth*$(G) > 0$. Otherwise we can find an element $w \in \mathcal{M}$ such that $\mathcal{M}^{j+1} = J\mathcal{M}^j + (w^{j+1})$ for every $j \ge t-1$. By using (12), we get for every $n = 2, \dots, p-1$

$$wI_n \subseteq \widetilde{\mathcal{M}^{n+1}} = \sum_{j=2}^{n+1} J^{n+1-j} I_j + \mathcal{M}^{n+1}$$

and

$$wI_p \subseteq \widetilde{\mathcal{M}^{p+1}} = J\widetilde{\mathcal{M}^p} = \sum_{j=2}^{p} J^{p+1-j} I_j + J\mathcal{M}^p \subseteq \sum_{j=2}^{p} J^{p+1-j} I_j + \mathcal{M}^{p+1}.$$

By applying the above proposition with $v = \sum_{i=2}^{p} v_i$, we can find an element $\sigma \in J\mathcal{M}^{v-1}$, such that for every $i = 1, \ldots, v_n$

$$w^v a_{in} \equiv \sigma a_{in} \mod \mathcal{M}^{v+n}. \tag{14}$$

On the other hand, since $\mathcal{M}^{k+1} \subseteq \widetilde{J\mathcal{M}^k} = \sum_{j=2}^{k} J^{k+1-j} I_j + J\mathcal{M}^k$, we can write

$$w^{k+1} = \sum_{j=2}^{k} \sum_{i=1}^{v_j} m_{ij} a_{ij} + b,$$

where $m_{ij} \in J^{k+1-j}$, $b \in J\mathcal{M}^k$. Using this and (14) we get

$$w^{v+k+1} = w^{k+1} w^v = \sum_{j=2}^{k} \sum_{i=1}^{v_j} m_{ij} a_{ij} w^v + b w^v = \sum_{j=2}^{k} \sum_{i=1}^{v_j} m_{ij}(\sigma a_{ij} + c) + b w^v,$$

where $c \in \mathcal{M}^{v+j}$. Since $j \le k$, $m_{ij} c \in J^{k+1-j} \mathcal{M}^{v+j} \subseteq J\mathcal{M}^{v+k}$ and $bw^v \in J\mathcal{M}^{v+k}$, so that

$$w^{v+k+1} = \sigma \sum_{j=2}^{k} \sum_{i=1}^{v_j} m_{ij} a_{ij} + q = \sigma(w^{k+1} - b) + q$$

with $q \in J\mathcal{M}^{v+k}$. Since $\sigma \in J\mathcal{M}^{v-1}$ and $w^{k+1} - b \in \mathcal{M}^{k+1}$, we get

$$w^{v+k+1} \in J\mathcal{M}^{v+k}$$

which, by Proposition 3.3, implies

$$\mathcal{M}^{v+k+1} = J\mathcal{M}^{v+k}.$$

We finally remark that by (13) and (11)

$$\begin{aligned}
v + k &= \sum_{i=2}^{p} v_i + k \le \sum_{i=2}^{t-1} (\rho_{i-1} - v_{i-1}) + \sum_{i=t}^{k} (\rho_{i-1} - 1) + \sum_{i=k+1}^{p} \rho_{i-1} + k \\
&\le \sum_{i \ge 1} \rho_i - (k - t + 1) + k - \sum_{i=1}^{t-2} v_i = e_1 - \rho_0 + t - 1 - \sum_{i=1}^{t-2} \left(e - \binom{h+i}{i} \right) \\
&= et - \binom{h+t}{h+1} + s - t - e + 1 + t - 1 - (t-2)e + \sum_{i=1}^{t-2} \binom{h+i}{i} \\
&= e - \binom{h+t}{h+1} + s + \binom{h+t-1}{t-2} - 1 = s.
\end{aligned}$$

Hence

$$\mathcal{M}^{s+1} = J\mathcal{M}^s$$

and the conclusion follows by Proposition 3.5. $\square$

We end the paper by giving suitable examples of Cohen–Macaulay local rings A such that $e=\binom{h+t-1}{h}+1$ and $indeg(A)\geq t$.

Let

$$A=k[[t^7,t^8+t^9,t^{12}]],$$

$$B=k[[t^7,t^8+t^9,t^{13}]],$$

$$C=k[[t^{11},t^{12}+t^{13},t^{18},t^{19}]].$$

Then we have

$$P_A(z)=\frac{1+2z+3z^2+z^4}{1-z},$$

$$P_B(z)=\frac{1+2z+3z^2+z^5}{1-z},$$

$$P_C(z)=\frac{1+3z+6z^2+z^6}{1-z}.$$

In all these examples $t=3$ and G is not Cohen–Macaulay, so that A has maximal Cohen–Macaulay type.

In the following class of examples A has maximal Cohen–Macaulay type and G is Cohen–Macaulay.

Let $t\geq 3$ and $I\subset R:=k[[X,Y,Z,T]]$ be the ideal generated by the maximal minors of the following $t\times(t+1)$ matrix, where if $t=3$ we only consider the first, the second and the last row:

$$\begin{pmatrix}
X & Z & 0 & 0 & . & . & . & . & . & . & 0\\
Y & X+Z^2 & T & 0 & . & . & . & . & . & . & 0\\
0 & 0 & Y & Z & 0 & . & . & . & . & . & 0\\
0 & 0 & 0 & Y & Z & 0 & . & . & . & . & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & . & . & . & . & . & 0 & Y & Z & 0\\
0 & 0 & . & . & . & . & . & . & 0 & T^2 & Z
\end{pmatrix}.$$

If we let $A=R/I$ then $indeg(A)=t$, $h=2$ and $e=\binom{t+1}{2}+1$.

The above computations have been carried over with the computer algebra program CoCoA.

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